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Schrödinger operators with random δ magnetic fields

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Abstract. We shall consider the Schrödinger operators on $\mathbf{R}^2$ with random δ magnetic fields. Under some mild conditions on the distribution of the random δ-fields, we prove the Lifshitz tail for our operators. The key of the proof is the Hardy type inequality by Laptev-Weidl [7].

1 Introduction

We consider the random magnetic Schrödinger operators on $\mathbf{R}^2$:

$$\mathcal{L}_\omega = \left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right)^2,$$

where ω is an element of some probability space $(\Omega, \mathbf{P})$. The vector-valued function $\mathbf{a}_\omega = (a_{\omega,x}, a_{\omega,y})$ is the magnetic vector potential, which corresponds to the magnetic field $\operatorname{rot} \mathbf{a}_\omega = \partial_x a_{\omega,y} - \partial_y a_{\omega,x}$. We assume

$$\operatorname{rot} \mathbf{a}_\omega(z) = \sum_{\gamma\in\Gamma_\omega} 2\pi\alpha_\gamma(\omega)\delta(z-\gamma) \tag{1}$$

in the distribution sense, where Γ_ω is a discrete set in $\mathbf{C}$, $\alpha(\omega) = \{\alpha_\gamma(\omega)\}_{\gamma\in\Gamma_\omega}$ are real numbers satisfying $0 \le \alpha_\gamma(\omega) < 1$, and δ is the Dirac measure concentrated on the origin. We consider the following assumptions for $(\Gamma_\omega, \alpha(\omega))$. In the sequel, we identify a vector $z = (x, y)$ with a complex number $z = x + iy$, and use notations $S + z = \{s + z \mid s \in S\}$ and $rS = \{rs \mid s \in S\}$ for $S \subset \mathbf{C}$, $z \in \mathbf{C}$ and $r > 0$.

Assumption 1.1 *(i) For any Borel set E in $\mathbf{R}^2$, the functions*

$$n_\omega(E) = \#(\Gamma_\omega \cap E), \quad \Phi_\omega(E) = \sum_{\gamma\in\Gamma_\omega\cap E} \alpha_\gamma(\omega)$$

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are measurable with respect to $\omega \in \Omega$.

(ii) *Let* $Q_0 = \{z = x+iy \mid -\frac{1}{2} \le x < \frac{1}{2},\ -\frac{1}{2} \le y < \frac{1}{2}\}$. *Then, for any Borel set* $E \subset Q_0$, *the random variables* $\{\Phi(E+n)\}_{n\in\mathbf{Z}\oplus i\mathbf{Z}}$ *are independently, identically distributed (abbrev. i.i.d.).*

(iii) *The mathematical expectation* $\mathbf{E}[\Phi(Q_0)]$ *is positive and finite, and the variance* $\mathbf{V}[\Phi(Q_0)]$ *is finite.*

(iv) *For any* $\epsilon > 0$, *the probability*

$$\mathbf{P}\{n(Q_0) \le 1 \textit{ and } \Phi(Q_0) < \epsilon\}$$

is positive.

(v) *For some* δ *with* $0 < \delta < 1$, *the probability*

$$\mathbf{P}\{n(Q_0) = n(\delta Q_0) = 1\}$$

is positive.

We can construct the vector potential $\mathbf{a}_\omega$ satisfying (1) by the following formula (see [5, section 4]):

$$\begin{aligned} \mathbf{a}_\omega &= (\operatorname{Im}\phi_\omega, \operatorname{Re}\phi_\omega), \\ \phi_\omega(z) &= \frac{\alpha_0(\omega)}{z} + \sum_{\gamma\in\Gamma_\omega\setminus\{0\}} \alpha_\gamma(\omega)\left(\frac{1}{z-\gamma} + \frac{1}{\gamma} + \frac{z}{\gamma^2}\right), \end{aligned} \tag{2}$$

where we put $\alpha_0(\omega) = 0$ if $0 \notin \Gamma$. We can prove that the sum in the above formula converges almost surely under (i), (ii) and (iii) of Assumption 1.1.

There are many examples satisfying Assumption 1.1. We list two typical examples below.

(i) **Perturbation of a lattice.** $\Gamma_\omega = \{n + f_n(\omega)\}_{n\in\mathbf{Z}\oplus i\mathbf{Z}}$, where $\{f_n\}$ are i.i.d., complex-valued random variables satisfying $|f_n(\omega)| < \delta/2$ for some deterministic constant δ with $0 < \delta < 1$. $\{\alpha_\gamma\}$ are $[0,1)$-valued i.i.d. random variables independent of $\{f_n\}$, satisfying $\mathbf{E}[\alpha_\gamma] > 0$ and

$$\mathbf{P}\{\alpha_\gamma < \epsilon\} > 0 \quad \text{for any } \epsilon > 0. \tag{3}$$

(ii) **Poisson model.** Γ_ω is a Poisson configuration (the support of the Poisson point process) on $\mathbf{C}$ with intensity measure $\rho dxdy$ for some positive constant ρ. $\{\alpha_\gamma\}$ are i.i.d. random variables independent of Γ_ω and satisfying $\mathbf{E}[\alpha_\gamma] > 0$ (the assumption (3) is not necessary).

For the definition of the Poisson point process, see [13, 2].

We denote the Friedrichs extension of $\mathcal{L}_\omega|_{C_0^\infty(\mathbf{R}^2\setminus\Gamma_\omega)}$ by H_ω. We can prove that the operator domain $D(H_\omega)$ of H_ω coincides with the functions in $L^2(\mathbf{R}^2)$ satisfying the boundary conditions

$$\mathcal{L}_\omega u \in L^2(\mathbf{R}^2), \qquad \lim_{z\to\gamma} |u(z)| < \infty \quad \text{for any } \gamma \in \Gamma_\omega. \tag{4}$$

Under (i)–(iv) of Assumption 1.1, we can prove

$$\sigma(H_\omega) = [0, \infty)$$

almost surely, by the usual method of approximating eigenfunctions (the technical detail will be given in our forthcoming paper [10]).

We shall introduce the integrated density of states (IDS) for the operator H_ω. Let $Q_k = \{z = x + iy \mid -k - \frac{1}{2} \le x < k + \frac{1}{2}, -k - \frac{1}{2} \le y < k + \frac{1}{2}\}$ for $k \ge 0$. Let H_ω^k be the self-adjoint realization of the operator $\mathcal{L}_\omega$ on $L^2(Q_k)$ with the Neumann boundary conditions $\left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right) u \cdot \mathbf{n} = 0$ on ∂Q_k ($\mathbf{n}$ is the unit outer normal). For $E \in \mathbf{R}$, we define

$$N_\omega^k(E) = \#\left\{\text{eigenvalues of } H_\omega^k \text{ less than or equal to } E\right\}, \tag{5}$$

$$N(E) = \lim_{k\to\infty} \frac{1}{|Q_k|} N_\omega^k(E), \tag{6}$$

where $|\cdot|$ denotes the Lebesgue measure. We can prove the limit $N(E)$ exists and independent of ω by Akcoglu-Krengel's superadditive ergodic theorem (see [4, 1]).

Our main result is the following inequality, called the *Lifshitz tail.*

Theorem 1.2 *Under (i)–(v) of Assumption 1.1, there exists some constant $C > 0$ and $E_0 > 0$ independent of ω and E, such that*

$$N(E) \le e^{-\frac{C}{E}} \tag{7}$$

for any E with $0 < E < E_0$.

There are numerous results which proved the Lifshitz tail for Schrödinger operators with random *scalar* potentials; see e.g. [4, 14]. There are also some results which proved the Lifshitz tail for Schrödinger operators with random *magnetic* fields; see Nakamura [11] and Klopp–Nakamura–Nakano–Nomura [6] for the discrete operators, Ueki [15], Nakamura [12], and Borg–Pulé [3] for the continuous operators. However, there seems to be no results for the Lifshitz tail for random δ magnetic fields, at present.

In Nakamura's paper [12], the crucial inequality in the proof of Lifshitz tail is Avron-Herbst-Simon estimate:

$$H_\omega \geq \operatorname{rot} \mathbf{a}_\omega. \tag{8}$$

If the magnetic field is regular, we can reduce the problem to the scalar potential case by using (8). However, in our case the inequality (8) is no longer useful, since $\operatorname{rot} \mathbf{a}_\omega = 0$ almost everywhere. Instead of (8), we use *the Hardy-type inequality* by Laptev–Weidl [7]. Below we sketch the main ingredient of the proof briefly.

2 Hardy-type inequality

For $d \geq 3$, there exists a positive constant C_d such that

$$\int_{\mathbf{R}^d} |\nabla u(x)|^2 dx \geq C_d \int_{\mathbf{R}^d} \frac{|u(x)|^2}{|x|^2} dx \tag{9}$$

for any $u \in C_0^\infty(\mathbf{R}^d)$. This inequality is called *the Hardy inequality.* The inequality (9) fails when $d = 2$, however, Laptev–Weidl [7] proved that a similar inequality holds if there exists a δ magnetic field at the origin.

Lemma 2.1 (Laptev-Weidl) *Let $\alpha \in \mathbf{R}$ and put $\mathbf{a}_\alpha(z) = \left(\operatorname{Im} \frac{\alpha}{z}, \operatorname{Re} \frac{\alpha}{z}\right)$ (so $\operatorname{rot} \mathbf{a}_\alpha = 2\pi\alpha\delta$). Then, we have*

$$\int_{|z|\leq R} \left| \left(\frac{1}{i}\nabla + \mathbf{a}_\alpha\right) u(z)\right|^2 dxdy \geq \rho(\alpha) \int_{|z|\leq R} \frac{|u(z)|^2}{|z|^2} dxdy \tag{10}$$

for any $R > 0$ and any $u \in C_0^\infty(\mathbf{R}^2 \setminus \{0\})$. Here $\rho(\alpha) = \min_{n\in\mathbf{Z}} |n + \alpha|^2$.

Proof. We use the polar coordinate $z = re^{i\theta}$. By a simple computation, we have

$$\left|\left(\frac{1}{i}\nabla + \mathbf{a}_\alpha\right) f(r)e^{in\theta}\right|^2 = |f'(r)|^2 + \frac{(n+\alpha)^2}{r^2}|f(r)|^2 \geq \frac{\rho(\alpha)}{r^2}|f(r)|^2.$$

So we get the conclusion by expanding u as a Fourier series with respect to θ. □

Let us return to our model. Let δ be the constant given in (v) of Assumption 1.1. Then, the probability of the event

$$n_\omega(Q_0 + n) = n_\omega(\delta Q_0 + n) = 1 \tag{11}$$

is positive for any $n \in \mathbf{Z} \oplus i\mathbf{Z}$. When (11) holds, we denote $\Gamma_\omega \cap (Q_0 + n) = \{\gamma_n(\omega)\}$, $\alpha_n(\omega) = \alpha_{\gamma_n(\omega)}(\omega)$. For $z \in n + Q_0$, define

$$V_\omega(z) = \begin{cases} \dfrac{4}{(1-\delta)^2}\rho(\alpha_n(\omega)) & \text{if (11) holds and } |z - \gamma_n(\omega)| < \dfrac{1-\delta}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

By using an appropriate gauge transformation and Lemma 2.1, we have

$$\int_{n+Q_0} \left|\left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right) u(z)\right|^2 dxdy \geq \rho(\alpha_n(\omega)) \int_{n+Q_0} V_\omega(z)|u(z)|^2 dxdy \tag{12}$$

for any $u \in C_0^\infty(\mathbf{R}^2 \setminus \Gamma_\omega)$.

Next, notice that[3]

$$\nabla|u| = \mathrm{Re}(\mathrm{sgn}\,\bar{u}\nabla u) = \mathrm{Re}(\mathrm{sgn}\,\bar{u}(\nabla + i\mathbf{a}_\omega)u) \qquad \text{a.e.} \tag{13}$$

holds for $u \in C_0^\infty(\mathbf{R}^2 \setminus \Gamma_\omega)$, where $\mathrm{sgn}\, z = z/|z|$ for $z \neq 0$ and $\mathrm{sgn}\, 0 = 0$. Taking the absolute value of the both sides, we have

$$\left|\left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right) u\right|^2 \geq \left|\nabla|u|\right|^2 \qquad \text{a.e.} \tag{14}$$

By (12) and (14), we have the following inequality:

Lemma 2.2

$$\int_{Q_k} \left|\left(\frac{1}{i}\nabla + \mathbf{a}_\omega\right) u\right|^2 dxdy \geq \frac{1}{2}\int_{Q_k} \left(\left|\nabla|u|\right|^2 + V_\omega\, |u|^2\right) dxdy \tag{15}$$

for any $u \in C_0^\infty(\mathbf{R}^2 \setminus \Gamma_\omega)$.

[3]The equality (13) holds for $u \in H^{1,1}_{\mathrm{loc}}(\mathbf{R}^2)$; see e.g. [8, appendix].

3 Outline of Proof of Theorem 1.2

By virtue of Lemma 2.2, we can reduce the problem to the scalar potential case, as we shall see below. The technical detail will be given in [10].

We use the following rough estimate for the eigenvalue counting functions:

$$N_\omega^k(E) \leq C_1|Q_k| \tag{16}$$

for any $E \leq 1$ and any $k = 1, 2, \ldots$, where C_1 is a constant independent of Γ_ω, $\alpha(\omega)$, E, and k. An inequality like (16) is well-known when the magnetic potential is smooth, and we can also prove (16) for our operator H_ω by using the diamagnetic inequality for Schrödinger operators with δ-magnetic fields [9].

It is known that

$$N(E) = \inf_{k\geq 1} \frac{1}{Q_k} \mathbf{E}\left[N_\omega^k(E)\right]$$

(see [4, VI.1.3]). Let $E_1(H_\omega^k)$ be the smallest eigenvalue of H_ω^k, and $\chi(\omega)$ the characteristic function of the event '$E_1(H_\omega^k) \leq E$'. Then we have for every $k \geq 1$ and $E \leq 1$

$$\begin{aligned} N(E) &\leq \frac{1}{|Q_k|}\mathbf{E}[N_\omega^k(E)] \\ &= \frac{1}{|Q_k|}\mathbf{E}[N_\omega^k(E)\chi(\omega)] \\ &\leq C_1\mathbf{P}\{E_1(H_\omega^k) \leq E\} \\ &\leq C_1\mathbf{P}\left\{E_1\left(\frac{1}{2}(-\Delta_N^k + V_\omega)\right) \leq E\right\}, \end{aligned} \tag{17}$$

where we used (16) in the second inequality, and Lemma 2.2 and the min-max principle in the last inequality. The potential V_ω is, roughly speaking, the Anderson-type scalar potential, [4] so we can use the well-known result (see e.g. [14, section 2.1])

$$\mathbf{P}\left\{E_1\left(\frac{1}{2}(-\Delta_N^k + V_\omega)\right) \leq E\right\} \leq e^{-CE^{-1}} \tag{18}$$

for sufficiently small $E > 0$. Thus we have the conclusion.

[4]The potential V_ω is not exactly the Anderson type scalar potential, since the 'center' of each single site potential varies a bit randomly from lattice points. However, the proof of the inequality (18) in [14] can be applied for our potential V_ω.

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